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A class of representations of affine Kac–Moody algebras

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Abstract. In one class of representations of simple affine algebras, there is a highest weight Λ, while all other dominant weights are of the form $\Lambda - q\delta$, where q is a positive integer and δ is a vector called the null vector. These representations are enumerated and a simple recursive formula is given for their weight multiplicities. The derivation of the formula illustrates some important features of affine algebras.

1. Introduction

Affine Kac-Moody algebras are important in many branches of modern physics, including string theories, conformal field theories and other field theories. In many models the physically realized irreps (irreducible representations) are of a simple type; they contain a highest weight Λ while all other contained dominant weights are of the form $\Lambda - q\delta$, where q is a positive integer and δ is a vector called the null vector. The purpose of this paper is to give a simple formula for finding all the weights in these irreps.

The basic properties of affine algebras have been given in the literature [1-3] and are summarized in section 2. A weight of a finite algebra of rank n may be characterized by n integral components, called here Dynkin components. A weight of an affine algebra of rank n is characterized by n integral Dynkin parameters and one other parameter, a null-depth parameter. For both finite and affine algebras a dominant weight is a weight with no negative Dynkin components.

It is convenient to organize the weights of a highest weight irrep in Weyl orbits, because the multiplicities of all weights in an orbit are the same in every irrep. Each orbit contains exactly one dominant weight, and is characterized by this weight. The problem of constructing an irrep may be separated into three steps. In the first, one finds the dominant weights in the irrep. In the second, one finds the multiplicities of these dominant weights. In the third, one finds the other weights in the orbit of each dominant weight, or alternately, finds a rule for identifying the orbit of each weight. Simple rules for the first and third steps for all highest weight irreps exist, and have been given in a previous reference [3]. We are concerned here only with the second step, finding the multiplicities.

No simple general formula exists for the multiplicities. This is clear because no such formula exists for finite algebras, which are subalgebras of affine algebras. Constructional procedures for finding these multiplicities do exist, the most useful one employing the Freudenthal recursion formula.

For each dominant set of Dynkin components in a highest weight irrep of an affine algebra, there is an infinite sequence of dominant weights, corresponding to increasing null depth. It is helpful to picture a change in null depth as a change in vertical

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position, and a change in components at the same null depth as a change in horizontal position. The width of an irrep, a finite number, may be defined as the number of dominant sets of Dynkin components in the irrep; this dimension is related to the sizes of irreps of finite algebras. Although the 'heights' of the affine irreps are infinite, in many cases the variation of multiplicity with null depth satisfies simple mathematical relations.

Although the quantities are defined precisely later, the general form of the main result will be given in this section. We are concerned with irreps of width one, called here narrow irreps. The contained dominant weights are $\Lambda - q\delta$, where δ is defined in section 2, and q ranges through all the non-negative integers. Let $W(q)$ be the multiplicity of $\Lambda - q\delta$. Then $W(0) = 1$. The other multiplicities may be determined successively from the formula,

$$W(q) = (1/q) \sum_{i=1}^{q} F_i W(q-i). \tag{1.1}$$

The integer F_i may be determined from the equation

$$F_i = \sum_j (jX_j) \tag{1.2}$$

where the sum over j is over all integral factors of i and the X_j are integers that depend on the irrep and the algebra, and are given in section 4. These X_j take on no more than two values for any of the irreps.

Other simple formulae for many irreps of affine algebras are known; many are given in terms of theta functions and other generating functions [4-6]. However, the formula given here has several attractive features. It is simple, and one can use it to calculate the multiplicities of a narrow irrep (down to some desired null depth) by hand in a small amount of time. If multiplicities for many irreps are desired, the formula is suitable for a small computer or programmable hand calculator.

The narrow irreps are listed in section 3. Section 5 contains a derivation of the multiplicity formula for many of the narrow irreps; it is not necessary to read this section to apply the formula. However, the derivation involves several cancellations, and reveals much about the structures of the algebras.

2. Basic properties of affine algebras

A simple affine algebra of rank n is represented by an indecomposable Coxeter-Dynkin diagram with n vertices, numbered here from 1 to n. Each vertex represents a simple root R_i. The generalized, n by n Cartan matrix A is defined by the scalar product equation,

$$A_{ij} = \langle R_i, R_j \rangle (1/R_j^2). \tag{2.1}$$

The n Dynkin components of a weight M are denoted by m_i and defined by

$$m_i = \langle M, R_i \rangle (2/R_i^2). \tag{2.2}$$

One may specify a weight of a highest weight irrep by the n Dynkin components and the displacement P_h, defined to be the number of times the simple root R_h is subtracted from the highest weight to obtain M. Any simple root may be chosen for R_h.

The determinant of A is zero for affine algebras. There is a null vector δ that satisfies the equation $\delta A = 0$, where δ is a row vector. The coefficients c_i of the δ vector are coefficients in an expansion in the simple roots, i.e.

$$\delta = \sum_i c_i R_i. \tag{2.3}$$

The vector δ is normalized by the condition that the c_i are as small as possible, consistent with all being positive integers. These c_i are called marks. Let α_i be a root such that $\alpha_i^2 \neq 0$. The co-root $\alpha_i^\vee$ is defined by the equation,

$$\alpha_i^\vee = \alpha_i(2/\alpha_i^2). \tag{2.4}$$

The co-marks $c_i^\vee$ are defined by writing δ as a linear combination of the simple co-roots, i.e. $\delta = \Sigma_i c_i^\vee R_i^\vee$. Therefore, $c_i^\vee = c_i R_i^2/2$.

The ratios of the lengths of the simple roots are given by the Cartan matrix. The overall root normalization is specified by the condition that the roots are as short as possible, consistent with the co-marks all being integers. The twist k of an algebra is given in terms of the longest simple root R_l by the formula $k = \frac{1}{2}R_l^2$. The Coxeter–Dynkin diagrams for the untwisted simple algebras are given in figure 1 and the diagrams for

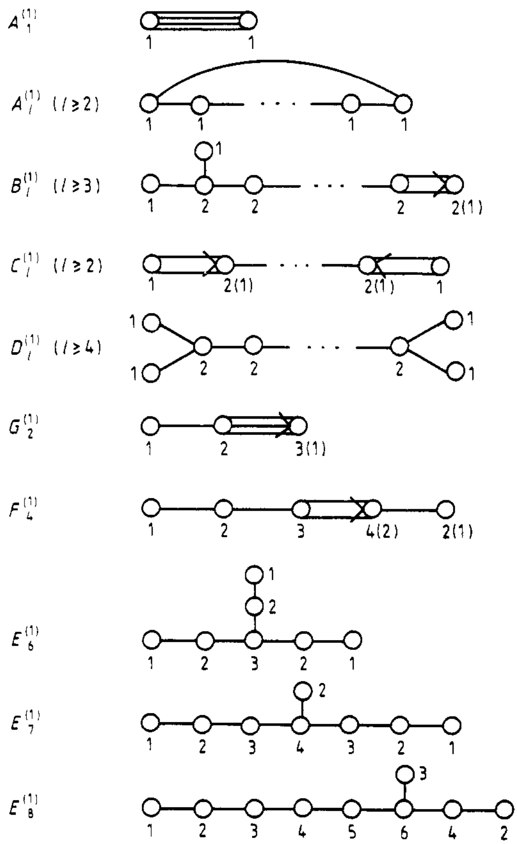

Figure 1. Diagrams for the untwisted affine algebras. An arrow points to the smaller of two connecting unequal roots, and the integer l is one less than the number of vertices. The first numbers by the vertices are the marks. If a co-mark is different from its mark, it is shown in parentheses after the mark.

the twisted algebras are given in figure 2. The superscript number in parentheses is the twist. The names of the algebras are taken from [1].

If a weight Q may be written as a linear combination of the simple co-roots, i.e. $Q=\Sigma_i Q_i^\vee R_i^\vee$, where the $Q_i^\vee$ are numerical coefficients, then it follows from the preceding arguments that if M is any weight,

$$\langle M, Q\rangle = \sum_i m_i Q_i^\vee . \tag{2.5}$$

The level L of a weight M is defined by the scalar product equation [2.3],

$$L(M) = \langle M, \delta\rangle = \sum_i m_i c_i^\vee . \tag{2.6}$$

The levels of all roots are zero and the levels of all weights in an irrep are the same. If we exclude the trivial zero weight, the level of every dominant weight is positive.

The vectors $j\delta$, where j is a positive or negative integer, are all roots, called imaginary roots. All other roots are called real roots.

The finite subalgebra $\mathscr{S}_h$ (called here a basic subalgebra) is obtained by ignoring the simple root R_h of the affine algebra, i.e. by ignoring the h vertex and connecting lines of the Coxeter–Dynkin diagram. The subweight with respect to S_h of an affine weight M is obtained by ignoring the Dynkin component m_h.

A scalar product may be defined for any two weights if one chooses one of the basic subalgebras $\mathscr{S}_h$. Then the scalar product is

$$\langle M, Q\rangle = -[P_h(M)L(Q)+P_h(Q)L(M)]c_h^{-1} + \langle M, Q\rangle_h \tag{2.7}$$

where $\langle M, Q\rangle_h$ is the scalar product of the subweights with respect to $\mathscr{S}_h$. If the weight Q is of level 0 it can be shown that this expression reduces to (2.5).

The null depth of a weight may be defined as P_h/c_h. The null depth is a convenient parameter when one compares weights with the same Dynkin components, since it is

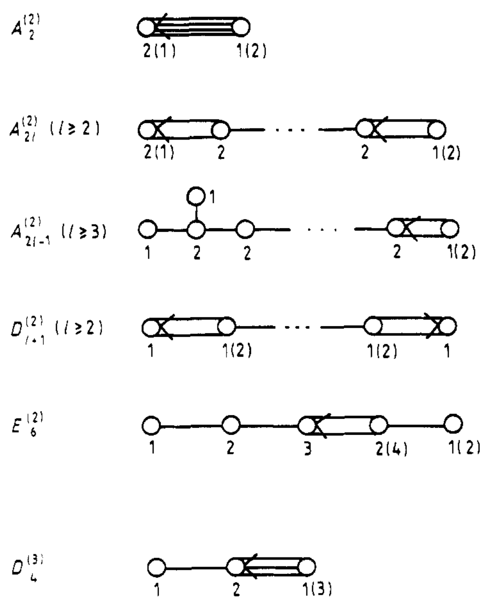

Figure 2. Diagrams for the twisted affine algebras. The notation is the same as that of figure 1.

independent of the choice of the index h. The displacement P_h is convenient when one compares weights with different Dynkin components, since it is always integral.

3. The narrow representations

The set of Dynkin components Λ' will appear in the irrep with highest weight Λ if and only if Λ and Λ' are congruent (in the same congruency class) [3]. Thus we must examine all the affine algebras and find all sets of dominant Dynkin components that are not congruent to any other set. Two sets of Dynkin components are congruent if and only if they satisfy two conditions: (i) They are on the same level, and (ii) the subweights with respect to any fundamental subalgebra are congruent, where a fundamental subalgebra is a basic subalgebra $\mathcal{S}_h$ such that the mark c_h is unity [3]. Application of this criterion is straightforward, but somewhat subtle, since both marks and co-marks are involved.

We examine first all the irreps of level 1. For this analysis the algebras of figures and 2 may be grouped into three classes:

Class I. The simply laced algebras. These are the $A_l^{(1)}$, $D_l^{(1)}$, and $E_l^{(1)}$ algebras, all untwisted.

Class II. The untwisted ($k=1$) algebras with non-zero roots of more than one length.

Class III. The twisted ($k=2$ or 3) algebras.

Examination of the diagrams of figures 1 and 2 shows that for all the algebras of classes I and III all level 1 irreps are narrow. Furthermore, when there are two or more such irreps of the same algebra, they are related by an obvious translation or rotation of the diagram, so their multiplicity structures are the same.

I illustrate by considering the algebra $E_6^{(1)}$, of figure 1. The roots are numbered according to the scheme

```
        7

        6

1   2   3   4   5
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There are three dominant level 1 weights, with sets of Dynkin components, $\Lambda_1=(1000000)$, $\Lambda_2=(0000100)$, and $\Lambda_3=(0000001)$, related by diagram rotations. Let us determine congruency by using the fundamental subalgebra $\mathcal{S}_7$. There are three triality classes for this E_6 algebra, corresponding to $m_1+2m_2+m_4+2m_5 \pmod 3$. Therefore, the classes of Λ_1, Λ_2, and Λ_3, respectively, are 1, 2, and 0. (In all the algebras of class I there is one level 1 irrep of each congruency class.)

The situation is quite different for the class II algebras, which are the algebras $B_l^{(1)}$, $C_l^{(1)}$, $G_2^{(1)}$, and $F_4^{(1)}$. Whereas every level 1 irrep of class I or III is narrow, every class II algebra contains two or more level 1 irreps that are congruent to each other, and so are not narrow. For each algebra the irrep (100 . . .) is such an irrep, where the first vertex is the left-hand vertex in figure 1.

In fact, there is only one level 1 series of narrow irreps of class II. If the right-hand (short) root of the diagram for $B_l^{(1)}(l \geqslant 3)$ is numbered l, then the irrep $\lambda_i=\delta_{il}$ (where λ_i are the Dynkin components of Λ) is narrow. This can be seen by determining congruency by means of the subalgebra $\mathcal{S}_1$, which corresponds to B_l. There are two congruency classes for B_l, given by m_l (mod 2). The irrep $\lambda_i = \delta_{il}$ is in the odd (spinor) class, while the other two level 1 irreps are in the even (tensor) class.

There is one other level 1 narrow irrep of class II, obtained by identifying the unit Dynkin component with the short root (middle vertex) of $C_2^{(1)}$. However, this algebra could have been called $B_2^{(1)}$, and will be called that here. Hence, the only narrow irreps of class II are those associated with the algebras $B_l^{(1)}(l \geq 2)$.

Next we consider levels greater than 1. An examination reveals that there are no narrow irreps with levels of 3 or greater, and only one series of level 2. This is the series $(10\ldots01)$ of $D_{l+1}^{(2)}(l \geq 2)$. The two unit Dynkin components correspond to the two shorter simple roots. If $\mathcal{S}_1$ is used to test congruency, this irrep is of the spinor class, while the $(l+1)$ other level-2 irreps are all of the tensor class.

The situation is similar to that of the $B^{(1)}$ series, in that there is one other algebra with a narrow level 2 irrep. The irrep is (11) of $A_1^{(1)}$. This algebra could be called $D_2^{(2)}$; the diagram is that which results when the last long root has been removed from between the shorter roots. I will refer to $A_1^{(1)}$ as the smallest member of the $D_{l+1}^{(2)}(l \geq 1)$ series. The algebra $A_1^{(1)}$ is the simplest of all affine algebras and is often discussed in the literature. It is interesting that some of its important properties are more closely related to the $D_{l+1}^{(2)}$ series than to the $A_l^{(1)}$ series.

4. The multiplicity formula for narrow irreps

The form of the multiplicity formula is given in (1.1) and (1.2); nothing more is needed except the values of the factors X_j. It can be shown that for all narrow level 1 irreps of algebras of classes I and III,

$$X_j = \mathcal{W}(j\delta) \tag{4.1}$$

where $\mathcal{W}(j\delta)$ is the multiplicity of the imaginary root $(j\delta)$. These root multiplicities have been tabulated [4, p 112]. If j/k is an integer then $\mathcal{W}(j\delta) = l$, one less than the rank of the algebra. For the twisted algebras $A_{2l}^{(2)}(l \geq 1)$, all the $\mathcal{W}(j\delta)$ are equal to l. For the other twisted algebras, if j/k is not an integer, the multiplicity is smaller, and is given by the equations,

$$\begin{aligned} &A_{2l-1}^{(2)}: && \mathcal{W}(j\delta) = l-1 && \text{for } j \text{ odd} \\ &D_{l+1}^{(2)}: && \mathcal{W}(j\delta) = 1 && \text{for } j \text{ odd} \\ &E_6^{(2)}: && \mathcal{W}(j\delta) = 2 && \text{for } j \text{ odd} \\ &D_4^{(3)}: && \mathcal{W}(j\delta) = 1 && \text{for } (j/3) \text{ non-integral.} \end{aligned} \tag{4.2}$$

For the narrow irreps of class I algebras and $A_{2l}^{(2)}$, (1.2) may be written in a simpler form. If J_i is the sum of all the integral factors of i $(J_i = \Sigma_j\, j)$ then

$$F_i = J_i l. \tag{4.3}$$

Next we turn to the narrow level 1 irreps of the class II algebras $B_l^{(1)}$. For these it can be shown that,

$$X_j(j \text{ odd}) = l+1 \qquad X_j(j \text{ even}) = l. \tag{4.4}$$

Finally, we consider the narrow level 2 series, of the algebras $D_{l+1}^{(2)}$, $l \geq 1$. For these the X_j are also given by (4.4).

I take as an example of the calculation the $q=6$ dominant weight of the narrow irrep of $D_4^{(3)}$. The integral factors of 6 are 1, 2, 3, and 6. From (4.1) the factor $X_j=2$ if j is a multiple of 3, and from (4.2), $X_j=1$ for other j. Therefore, from (1.2), $F_6=2(3+6)+(1+2)=21$. In a similar manner one finds that the F_i for $i=1$ to 5, are, respectively, 1, 3, 7, 7, and 6. Suppose one has calculated the multiplicities $W(q)$ for $q=1$ to 5; they are 1, 2, 4, 6, and 9 respectively. Then, from (1.1),

$$W(6)=\tfrac{1}{6}[W(5)+3W(4)+7W(3)+7W(2)+6W(1)+21W(0)]=16.$$

5. Derivation of the multiplicity formula

The results of (1.1), (1.2), (4.1) and (4.4) are based on the Freudenthal recursion formula, in which the multiplicity $W(M)$ of the weight M in the irrep Λ is determined from the multiplicities of higher weights by the equations [2, equations (23)],

$$W(M)=N/D \tag{5.1}$$

$$N=\sum_{\alpha>0}\sum_{p=1}^{\infty}\langle M+p\alpha,\alpha\rangle W(M+p\alpha) \tag{5.2}$$

$$D=\tfrac{1}{2}\langle\Lambda+M+2S,\Lambda-M\rangle \tag{5.3}$$

where the sum over α is over all positive roots (both real and imaginary) and S is the weight such that every Dynkin component is unity. We need to consider only weights M that are dominant, and so of the form $\Lambda-q\delta$.

Before applying the formula, we establish some preliminary results. The multiplicities of the positive real roots are all one. However, since there are an infinite number of these roots, we organize them in layers and families, as was done in a previous reference [7]. The root structure is based on the well-known fact,

$$\text{if } \alpha \text{ is a root, then } \alpha+rk\,\delta \text{ is a root} \tag{5.4}$$

for all integers r. Each real root belongs to a specific layer. For the roots α of the $(j+1)$ layer, the co-root-basis components of $\alpha-jk\delta$ are all non-negative and the co-root-basis components of $(j+1)k\delta-\alpha$ are all non-negative. The real roots may all be written in the form

$$\alpha_a+jk\delta \tag{5.5}$$

where α_a is a root of the first layer and j ranges through all the integers. The positive real roots correspond to non-negative values of j. A family includes the roots for a particular α_a and all values of j. For every real root α there is a different, partner root, defined to be the root in the layer of α and the family of $(-\alpha)$.

Roots are called short, normal, or overlong, if their norms are less than 2, equal to 2, or greater than 2, respectively. Overlong roots exist only if the twist k is 2 or 3. It is known that if the k in the recursion relation of (5.4) were replaced by 1, the relation would still apply to all roots except the overlong roots. Therefore, for the short and normal roots each layer may be separated into k minilayers. For these we remove the k in (5.5). The set of roots $\alpha_a+j\delta$, where α_a is in the first minilayer, represents a double or triple family (if $k=2$ or 3). The minipartner of a short or normal root α is in the minilayer of α. If α and α' are minipartners in the first minilayer, then $\alpha+\alpha'=\delta$.

For any of the affine algebras, let ν_s and ν_n be the numbers of short and normal roots in a minilayer, and ν_0 be the number of overlong roots in a layer. The values of these numbers for the algebras of classes II and III that contain narrow irreps are given in table 1.

Next, we need a simple method of identifying the orbit of the $M+p\alpha$ of (5.2). The norm may be used for this. We consider the norm difference $\langle M, M\rangle - \langle\Lambda, \Lambda\rangle$. If a weight is the type $\Lambda - q\delta$, then the Dynkin components of M and Λ are the same, so it follows from (2.7) that $\langle M, M\rangle - \langle\Lambda, \Lambda\rangle = -2P_hL/c_h$. Each time δ is subtracted, P_h is increased by c_h, so we may write

$$\langle M, M\rangle - \langle\Lambda, \Lambda\rangle = -2Lq. \tag{5.6}$$

Since the dominant weight of every orbit is of the type $\Lambda - q\delta$, and since the norm is invariant to Weyl reflections, (5.6) may be used to identify the orbit q of any weight in the irrep. The quantity of interest is $\Delta q = q(M) - q(M+p\alpha)$. This is given by

$$\Delta q = (\langle M+p\alpha, M+p\alpha\rangle - \langle M, M\rangle)/(2L). \tag{5.7}$$

The Coxeter number C and the dual Coxeter number $C^\vee$ are defined for a simple affine algebra by the equations,

$$C = \sum_i c_i \qquad C^\vee = \sum_i c_i^\vee.$$

It is pointed out in [7] that for all the simple affine algebras, the total number ν of real roots in a layer $[\nu = k(\nu_s + \nu_n) + \nu_0]$ satisfies the equation

$$\nu = lkC. \tag{5.8}$$

We now limit attention to the twisted algebras with no short roots, the last four algebras of table 1. If $\mathscr{W}(q\delta)$ when q/k is not integral is denoted by w, and this number is taken from (4.2), the layer sizes are taken from table 1, and the dual Coxeter numbers are taken from figure 2, it is seen that two additional relations are satisfied for these four algebras,

$$\nu_n = wC^\vee \tag{5.9}$$

$$\nu_0 = (l-w)C^\vee. \tag{5.10}$$

We are now ready to apply the Freudenthal formula to the weight $M = \Lambda - q\delta$. For definiteness we consider the three special cases of the level 1 irreps of the $k=2$ algebras with no short roots. In the denominator, (5.3), $\Lambda - M = q\delta$. From (2.6), $\langle\Lambda + M, q\delta\rangle = 2q$ and $\langle S, q\delta\rangle = qC^\vee$. Therefore, the denominator of (5.3) is

$$D = q(1 + C^\vee). \tag{5.11}$$

Table 1. Numbers of short and normal roots in a minilayer and overlong roots in a layer for affine algebras of classes II and III that have narrow irreps.

Algebra	ν_s	ν_n	ν_0
$B_l^{(1)}$	$2l$	$2l(l-1)$	0
$A_{2l}^{(2)}$	$2l$	$2l(l-1)$	$2l$
$A_{2l-1}^{(2)}$	0	$2l(l-1)$	$2l$
$D_{l+1}^{(2)}$	0	$2l$	$2l(l-1)$
$E_6^{(2)}$	0	24	24
$D_4^{(3)}$	0	6	6

The numerator N of (5.2) may be written as a sum of two terms, $N = N^{\mathrm{im}} + N^{\mathrm{re}}$, representing the contributions of imaginary and real roots to the sum over α. We consider N^{im} first. Let $\alpha = j\delta$, where j is a positive integer. Then,

$$\langle M + pj\delta, j\delta\rangle = j\langle M, \delta\rangle = j \tag{5.12}$$

since the level of M is 1. The change in null depth q between $M + pj\delta$ and M is pj. Therefore each imaginary root $j\delta$ contributes a factor of j whenever j is an integral factor of the change in null depth. There are $\mathcal{W}(j\delta)$ imaginary roots j. Therefore, N^{im} is given by

$$N^{\mathrm{im}} = \sum_{i=1}^{q} \sum_{j} [j\mathcal{W}(j\delta)]\, W(q-i) \tag{5.13}$$

where the sum over j is over integral factors of i. It is convenient to replace $\mathcal{W}(j\delta)$ by $w + [\mathcal{W}(j\delta) - w]$, where $w = \mathcal{W}(\delta)$ (4.2). Since $k = 2$, $\mathcal{W}(j\delta) - w$ is 0 for odd j and $(l - w)$ for even j. Then N^{im} is written as a sum of two pieces,

$$N^{\mathrm{im}} = N_1^{\mathrm{im}} + N_2^{\mathrm{im}} \tag{5.14}$$

$$N_1^{\mathrm{im}} = w \sum_{i=1}^{q} J_i W(q-i) \tag{5.15a}$$

$$N_2^{\mathrm{im}} = (l - w) \sum_{i} J_{i,\mathrm{even}} W(q-i) \tag{5.15b}$$

where J_i is the sum of integral factors of i and $J_{i,\mathrm{even}}$ is the sum of even integral factors of i.

Since, for the cases under consideration the factors X_j of (1.2) are equal to $\mathcal{W}(j\delta)$, an analogous separation of the final result of (1.1) may be made. Thus, $W(q) = W_1 + W_2$, where

$$W_1 = (1/q) w \sum_{i=1}^{q} J_i W(q-i) \tag{5.16a}$$

$$W_2 = (1/q)(l - w) \sum_{i} J_{i,\mathrm{even}} W(q-i). \tag{5.16b}$$

We now turn to the contributions of the real roots. The contributions of the normal real roots and overlong real roots are called N_1^{re} and N_2^{re}, respectively. It is convenient to write the function J_i as a sum of terms, each containing one or two integral factors of i, according to the following rules. Let

$$J_i = \sum_{p} J_{p,i} \tag{5.17}$$

where p runs through the integral factors of i that are not larger than $(i)^{1/2}$. The rules are,

$$\text{if } p = (i)^{1/2}, \text{ then } J_{p,i} = p \tag{5.18a}$$

$$\text{if } p < (i)^{1/2}, \text{ then } J_{p,i} = p + (i/p). \tag{5.18b}$$

For example, if $i = 9$, p takes the values 1 and 3, $J_{1,9} = 1 + 9 = 10$ and $J_{3,9} = 3$.

In accordance with the discussion below (5.5), the normal root α is written in the form $\alpha_a + j\delta$, where α_a is in the first minilayer and j is a non-negative integer. Then the scalar product (SP) of (5.2) is $\mathrm{SP} = \langle M + p\alpha_a + pj\delta,\ \alpha_a + j\delta\rangle$. Equation (2.5) is convenient for evaluating the scalar product $\langle M, \alpha_a\rangle$ while the level equation (2.6) may be used for scalar products involving δ. We take the unit Dynkin component of M to be the first component. Since M is of level 1 and $\alpha_a^2=2$,

$$\mathrm{SP}=\alpha_a^\vee(1)+j+2p \tag{5.19a}$$

where $\alpha_a^\vee(1)$ is the first co-root basis element of α_a.

The change in null depth, which is the i of (1.1), may be calculated from (5.7). The result is

$$i=p\alpha_a^\vee(1)+pj+p^2. \tag{5.19b}$$

If j' is defined by the equation $j'=j+\alpha_a^\vee(1)$, the last two equations may be written,

$$\mathrm{SP}=2p+j' \tag{5.20a}$$

$$i=p(p+j'). \tag{5.20b}$$

Since each pair of minipartners α_a and α'_a satisfy $\alpha_a+\alpha'_a=\delta$ and since $c_1^\vee=1$, the component $\alpha_a^\vee(1)$ is 0 for exactly half the roots α_a and 1 for the other half. It is seen from (5.20a, b) that the scalar products and i values from these two sets of root double-families are the same, except that j' starts with 0 for the $\alpha_a^\vee(1)=0$ terms and starts with 1 for the $\alpha_a^\vee(1)=1$ terms.

Let us consider together all the terms for a fixed p. The smallest value of i is p^2, corresponding to $j'=0$. For this term only half the families contribute; the contribution to the N of (5.2) is

$$\nu_n pW(q-p^2)=\nu_n pW(q-i) \tag{5.21}$$

where $i=p^2$. Next we consider the terms $j'>0$ in (5.20a, b). All families contribute to these terms. The contribution is

$$\nu_n\sum_{j'>0}(2p+j')W[q-p(p+j')]=\nu_n\sum_{i>p^2}[p+(i/p)]W(q-i) \tag{5.22}$$

where the last sum includes all $i>p^2$ such that i/p is an integer. The total contribution of all terms with a fixed p is

$$\nu_n\sum_i J_{p,i}W(q-i) \tag{5.23}$$

where $J_{p,i}$ is the function of (5.18a, b). If the contributions from all values of p are added, it is seen from (5.17) that the result is

$$N_1^{\mathrm{re}}=\nu_n\sum_i J_i W(q-i). \tag{5.24}$$

If one adds this to the N_1^{im} of (5.15a), the result is $N_1=N_1^{\mathrm{im}}+N_1^{\mathrm{re}}=(w+\nu_n)\sum_i J_i W(q-i)$. If (5.9) is used,

$$N_1=w(1+C^\vee)\sum_i J_i W(q-i). \tag{5.25}$$

The factor $(1+C^\vee)$ cancels with that in D, so that

$$W_1(q)=N_1/D=(1/q)w\sum_i J_i W(q-i). \tag{5.26}$$

We next consider N_2^{re}, the contributions of the overlong roots. As in (5.5) each positive overlong root α is written in the form $\alpha=\alpha_a+2j\delta$, where α_a is in the first layer and j is a non-negative integer. If the same procedure used for the normal roots is followed, the equations analogous to (5.19a, b) are

$$\mathrm{SP} = \alpha_a^\vee(1) + 2j + 4p$$

$$i = p\alpha_a^\vee(1) + 2pj + 2p^2.$$

In this case the quantity $j + \frac{1}{2}\alpha_a^\vee(1)$ is called j', so the equations may be written,

$$\mathrm{SP} = 4p + 2j' \tag{5.27a}$$

$$i = 2p(p + j'). \tag{5.27b}$$

All overlong roots may be generated from the simple overlong roots by simple Weyl reflections. It can be shown from this fact and from the general structure of the Cartan matrix that all co-root-basis elements of overlong roots are integral multiples of the twist, in this case 2. Since $c_1^\vee = 1$, and since an overlong root in the first layer and its partner satisfy $\alpha + \alpha' = 2\delta$, it follows that $\alpha_a^\vee(1)$ is 0 for half the overlong roots and 2 for the other half. Therefore j' is either j or $j+1$. If (5.27a, b) are compared to (5.20a, b) it is seen that for fixed p and j', the scalar product and change in null depth are just twice the corresponding values for the normal roots. We have found earlier that the average contribution of a normal root double-family to the null depth $(q-i)$ is J_i. It is clear that the average contribution of an overlong root family is twice the sum of the integral factors of $(\frac{1}{2}i)$. This is just the sum of the even integral factors of i. Therefore,

$$N_2^{\mathrm{re}} = \nu_0 \sum_i J_{i,\mathrm{even}} W(q-i).$$

If this is added to N_2^{im} the result is

$$N_2 = [(l-w) + \nu_0] \sum_i J_{i,\mathrm{even}} W(q-i). \tag{5.28}$$

If use is made of (5.10), $l - w + \nu_0 = (l-w)(1 + C^\vee)$. Again the $(1 + C^\vee)$ factor cancels that of (5.11), so

$$W_2 = N_2/D = (1/q)(l-w) \sum_i J_{i,\mathrm{even}} W(q-i). \tag{5.29}$$

The W_1 and W_2 of (5.26) and (5.29) are equal to those of (5.16a, b), proving the result.

In order to extend the above proof to the narrow representation of $D_4^{(3)}$, one has only to make the obvious modifications necessary for a twist-3 algebra. The above proof includes as special cases the narrow irreps of the class I algebras $A_l^{(1)}$, $D_l^{(1)}$ and $E_l^{(1)}$. In these cases there are no N_2^{im} and N_2^{re}. The entire contribution of the imaginary roots is called N_1^{im}, and the w in (5.26) is replaced by l. Equation (5.8) is used to replace the factor $\nu_n = \nu$ in N_1^{re} by $lC^\vee$. (Recall that $C = C^\vee$ for simply laced algebras).

The same general techniques may be used to establish the results for the other cases, the level 1 narrow irreps of $A_{2l}^{(2)}$ and $B_l^{(1)}$ and the level 2 narrow irreps of $D_{l+1}^{(2)}$. In these cases one does not use (5.9) and (5.10) but writes the quantities in terms of l, using table 1. A few comments will be made about the contributions of the short roots to the irreps of $A_{2l}^{(2)}$ and $B_l^{(1)}$. The co-root basis component $\alpha_a^\vee(1)$ of every short root in the first minilayer is $\frac{1}{2}$. The total contribution of the short roots to the numerator N of (5.2) is

$$\nu_s \sum_i [\tfrac{3}{2}J_{i,\mathrm{odd}} + \tfrac{1}{2}J_{i,\mathrm{even}}] W(q-i)$$

where $J_{i,\mathrm{odd}}$ is the sum of all odd integral factors of i.

6. Concluding remarks

The narrow irreps of simple affine algebras are all the level 1 irreps of the simply laced algebras and the twisted algebras, the level 1 irrep of $B_l^{(1)}(l \geqslant 2)$ in which the unit Dynkin component is associated with the short simple root and the level 2 irrep of $D_{l+1}^{(2)}(l \geqslant 1)$ in which unit Dynkin components are associated with the two normal roots. The multiplicities for the dominant weights of these irreps are determined from (1.1) and (1.2), with the X_j factors given in section 4.

For these representations the multiplicities of non-dominant weights follow simple rules. Let M be a non-dominant set of Dynkin components that is congruent to Λ. Then, if one selects a convenient simple root R_h, there is a weight M_0 of displacement P_h and components M that is in the top orbit, the orbit of Λ. The value of P_h may be determined from the condition that the norms of M_0 and Λ are equal. Each orbit q will then contain the weight $M_0 - q\delta$, with multiplicity $W(q)$.

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